Exam 2 Practice

April 11, 2015

Exercise 1

For a random variable X, the expected value of the random variable $3X$ is

$$E(3X) = 3E(X).$$

(a) TRUE
(b) FALSE

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$$E(3X) = 3E(X).$$

(a) TRUE
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Exercise 2

For a random variable X, the expected value of the random variable X^3 is

$$E(X^3) = E(X)^3.$$

(a) TRUE
(b) FALSE

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For a random variable X, the expected value of the random variable X^3 is

$$E(X^3) = E(X)^3.$$

(a) TRUE
(b) FALSE

In general, $E(g(X)) \neq g(E(X))$. That only works in the special case of linear functions $g(x) = ax + b$.

Exercise 3, part 1

The discrete random variable X has the following probability mass function :

$$P(X = 1) = 2/5, \; P(X = 2) = 2/5, \; P(X = 3) = 0, \; P(X = 4) = 1/5.$$

Find the expectation and variance of X.

Exercise 3, part 1

(a) $E(X) = 2; \quad var(X) = \frac{2}{5}$

(b) $E(X) = \frac{2}{5}; \quad var(X) = \frac{6}{5}$

(c) $E(X) = 2; \quad var(X) = \frac{2}{5}$

(d) $E(X) = 2; \quad var(X) = \frac{6}{5}$

(e) $E(X) = \frac{2}{5}; \quad var(X) = 2$

Exercise 3 part 1,

(a) $E(X) = 2; \quad var(X) = \frac{2}{5}$

(b) $E(X) = \frac{2}{5}; \quad var(X) = \frac{6}{5}$

(c) $E(X) = 2; \quad var(X) = \frac{2}{5}$

(d) $E(X) = 2; \quad var(X) = \frac{6}{5}$

(e) $E(X) = \frac{2}{5}; \quad var(X) = 2$

Exercise 3, part 1, solution

The expectation is

$$EX = 1 \times 2/5 + 2 \times 2/5 + 4 \times 1/5 = 2.$$

The variance is

$$\mathrm{Var}\, X = (1-2)^2 \times 2/5 + (2-2)^2 \times 2/5 + (4-2)^2 \times 1/5 = 6/5$$

Exercise 1, part 2

A discrete random variable X has the following probability mass function:

$$P(X = 1) = 2/5,\ P(X = 2) = 2/5,\ P(X = 3) = 0,\ P(X = 4) = 1/5.$$

We measure X three times independently. Find the probability that $X = 4$ for exactly two of those measurements.

Exercise 3, part 2

The probability that $X = 4$ for exactly two of those measurements is

(a) $\dfrac{2}{25}$

(b) $\dfrac{12}{125}$

(c) $\dfrac{12}{25}$

(d) $\dfrac{3}{2}$

(e) $\dfrac{3}{5}$

Exercise 3, part 2

The probability that $X = 4$ for exactly two of those measurements is

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(b) $\frac{12}{125}$

(c) $\frac{12}{25}$

(d) $\frac{3}{2}$

(e) $\frac{3}{5}$

Exercise 3, part 2, solution

The number of times you get 4 from three measurements is binomially distributed with $n = 3$ and $p = P(X = 4) = 1/5$. Therefore the probability of getting 4 exactly twice is

$$\binom{3}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right) = \frac{12}{125}.$$

Exercise 4, part 1

A continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{2}{x^2} & \text{for } 1 \le x \le 2; \\ 0 & \text{otherwise.} \end{cases}$$

1. Find the cumulative distribution function $F(x) = P(X \le x)$.
2. Find the value x such that $P(X \le x) = P(X \ge x) = 0.5$.

Exercise 4, part 1

The value x such that $P(X \leq x) = P(X \geq x) = 0.5$ is

(a) $\frac{4}{3}$

(b) $\frac{2}{3}$

(c) -2

(d) 0

(e) $-\frac{3}{2}$

Exercise 4, part 1

The value x such that $P(X \leq x) = P(X \geq x) = 0.5$ is

(a) $\frac{4}{3}$

(b) $\frac{2}{3}$

(c) -2

(d) 0

(e) $-\frac{3}{2}$

Exercise 4, part 1, solution

1. Note that the range of possible values for X is $1 \leq X \leq 2$.

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \begin{cases} \int_{-\infty}^{x} f(u)\,du = 0 & ; x \leq 1 \\ \int_{1}^{x} \frac{2}{u^2}\,du = -2/u\big|_1^x = \frac{-2}{x} + 2 & ; 1 < x < 2 \\ \int_{1}^{x} f(u)\,du = \int_{1}^{2} \frac{2}{u^2}\,du = 1 & ; x \geq 2 \end{cases}$$

2. We need $-2/x + 2 = 0.5$, so $-2/x = -1.5$, so $x = 2/1.5 = 4/3$.

Exercise 4, part 2

Calculate the expectation of the random variable $\dfrac{X}{2} - \ln 2$.

Recall, X has probability density function given by

$$f(x) = \begin{cases} \frac{2}{x^2} & \text{for } 1 \leq x \leq 2; \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 4, part 2

The expected value $E(\frac{X}{2} - \ln 2)$ is

(a) $\ln 2$

(b) $\frac{2}{\ln 2}$

(c) 2

(d) 0

(e) $-\frac{\ln 2}{2}$

Exercise 4, part 2

The expected value $E(\frac{X}{2} - \ln 2)$ is

(a) $\ln 2$

(b) $\frac{2}{\ln 2}$

(c) 2

(d) 0

(e) $-\frac{\ln 2}{2}$

Exercise 4, part 2, solution

Note that since $f(x) = 0$ outside of the interval $[1, 2]$,

$$\int_{-\infty}^{\infty} xf(x)\, dx = \int_{1}^{2} x\frac{2}{x^2}.$$

The expectation of X is

$$EX = \int_{-\infty}^{\infty} x\frac{2}{x^2}\, dx = \int_{1}^{2} x\frac{2}{x^2}\, dx$$

$$= \int_{1}^{2} \frac{2}{x}\, dx = [2\ln|x|]_1^2 = 2\ln 2 - 2\ln 1 = 2\ln 2.$$

Therefore,

$$E(\frac{X}{2} - \ln 2) = \frac{E(X)}{2} - \ln 2 = \frac{2\ln 2}{2} - \ln 2 = 0.$$

Exercise 5

Suppose a continuous random variable X has probability density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Find E(X).

Exercise 5

The expected value $E(X)$ is

(a) $\dfrac{1}{2}$

(b) $\dfrac{-1}{2}$

(c) 1

(d) 0

(e) $\dfrac{1}{2}$

Exercise 5

The expected value $E(X)$ is

(a) $\frac{1}{2}$

(b) $\frac{-1}{2}$

(c) 1

(d) 0

(e) $\frac{1}{2}$

Exercise 5, solution

$$E(X) = \int_{-\infty}^{\infty} 2xe^{-2x} dx = \int_{0}^{\infty} 2xe^{-2x} dx$$

since $f(x) = 0$ for $x \leq 0$. Thus,

$$E(X) = \lim_{b\to\infty} \int_0^b 2xe^{-2x} dx = \lim_{b\to\infty} (-xe^{-2x} - \frac{1}{2}e^{-2x})\Big|_0^b$$

$$= \lim_{b\to\infty} (-be^{-2b} - \frac{1}{2}e^{-2b} + \frac{1}{2}) = \frac{1}{2} - \lim_{b\to\infty} be^{-2b} - \frac{1}{2} \lim_{b\to\infty} e^{-2b}.$$

By L'Hopital,

$$\lim_{b\to\infty} be^{-2b} = \lim_{b\to\infty} \frac{b}{e^{2b}} \overset{\frac{\infty}{\infty}}{=} \lim_{b\to\infty} \frac{1}{2e^{2b}} = 0.$$

Also,

$$\lim_{b\to\infty} e^{-2b} = 0, \text{ so } E(X) = \frac{1}{2}.$$

Exercise 6

What is the solution to the differential equation

$$\frac{dy}{dx} = 5(y - 1224234)(y + 3533),$$

subject to the initial value condition $y(6) = -3533$?

Exercise 6

The solution to the differential equation

$$\frac{dy}{dx} = 5(y - 1224234)(y + 3533),$$

subject to the initial value condition $y(6) = -3533$ is

(a) $y(x) = 465376e^{3533x}$

(b) $y(x) = 3533$

(c) $y(x) = -3533$

(d) $y(x) = -\frac{3533}{6}x$

(e) There is no such solution.

Exercise 6

The solution to the differential equation

$$\frac{dy}{dx} = 5(y - 1224234)(y + 3533),$$

subject to the initial value condition $y(6) = 3533$ is

(a) $y(x) = 465376e^{3533x}$

(b) $y(x) = 3533$

(c) $y(x) = -3533$

(d) $y(x) = -\frac{3533}{6}x$

(e) There is no such solution.

Do not forget about the trivial solutions (the constant functions that make $\frac{dy}{dx} = 0$. !! Those are always part of the general solution.

Exercise 7

Find a 2x2 matrix A such that A transforms an arbitrary vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{A} \begin{pmatrix} y \\ -x \end{pmatrix},$$

namely a reflection across the $x = y$ axis and a reflection across the x-axis.

Exercise 7

The matrix A=

(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(e) There is no such matrix.

Exercise 7

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(d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(e) There is no such matrix.

Exercise 7, solution

Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

So

$$ax + by = y \text{ for ALL } x, y$$

and

$$cx + dy = -x \text{ for ALL } x, y.$$

Thus $a = 0, b = 1, c = -1, d = 0$.

Exercise 8

The vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$

(a) TRUE

(b) FALSE

Exercise 8, solution

The vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$

(a) TRUE

(b) FALSE

We can check

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 2.

Exercise 9

1. Find the inverse of the matrix (if it exists)

$$\begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$$

2. Solve the following matrix equation

$$\begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Exercise 9

The solution to the matrix equation is

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(d) There are infinitely many solutions.

(e) There is no solution.

Exercise 9

The solution to the matrix equation is

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(d) There are infinitely many solutions.

(e) There is no solution.

Exercise 9, solution

1. The determinant is $0 \times 1 - 2 \times (-1) = 2$, so the inverse matrix is

$$\frac{1}{2}\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & -1 \\ 1/2 & 0 \end{pmatrix}.$$

2. Since the matrix on the left-hand side has an inverse, we multiply by that inverse to get the solution:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & -1 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \end{aligned}$$

Therefore $x = 0$ and $y = 2$.

Exercise 10

Find values for the constants a, b such that the matrix equation

$$\begin{pmatrix} 0 & a \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ 2 \end{pmatrix}$$

has

1. exactly one solution;
2. infinitely many solutions;
3. no solutions.

(Note: the three cases are separate – you will find a different pair of values for a, b in each case.)

Exercise 10, solution

1. To get exactly one solution, we need to choose a such that the determinant of the matrix on the left-hand side is not zero. Any a not equal to 0 will do, and any value of b will do. So $a \in \mathbb{R} \backslash \{0\}$ and $b \in \mathbb{R}$.
2. To get infinitely many solutions, we need to choose a such that the determinant of the matrix *is* zero, and $a = 0$ does this. We then need to make sure there is at least one solution which will happen only when $b = 0$. So $a = 0$ and $b = 0$ is the answer.
3. To get no solutions, we need $a = 0$ as before, and then any b not equal to zero will work. So $a = 0$ and $b \in \mathbb{R} \backslash \{0\}$.

Also make sure to review

- how to solve a differential equation step by step to find the general solution or the particular solution corresponding to an initial value constraint (note that the trivial solutions are always part of the general solution)
- what it means by definition for a number to be an eigenvalue and how to find the eigenvalues of a matrix going step by step straight from the definition
- how to find the set of eigenvectors corresponding to an eigenvalue

Advice: review all the notes and homework problems.